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ON TRANSIENT CREEP BOUNDS FOR SAINT-VENANT PURE BENDING PROBLEMS†

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Abstract—The field equations governing primary and secondary creep, with the inclusion of elastic strains, in the Saint-Venant theory of pure bending are reduced to a single equation in the tensile stress. It is shown how, from this equation, the limiting or steady-state stresses can be obtained. This equation is then used to derive inequalities which describe the shape of the stress profile at all times, and from which bounds on stresses and displacements are easily obtained. The inequalities, which are similar to those of [1], are established using new arguments which greatly simplify the analysis for the case of primary creep.

1. INTRODUCTION

In [1], the field equations governing primary creep in spherical and incompressible cylindrical pressure vessels subject to a nondecreasing internal pressure were reduced to a single equation ((2.34) of [1]) in the effective stress. From this equation, *a priori* bounds were obtained for the effective stress which were shown to imply simple, easily computed bounds for various quantities of physical interest. It was also shown that the equation could be used to formally derive the large time limit of the stresses, e.g. to recover Hult's results (33) and (34) of [2].

In the present paper, results analogous to the above are obtained for a class of problems whose geometry is quite different. That is, we consider a bisymmetrical beam whose lateral surface is stress-free and whose ends are subject to a monotone bending moment. It is required that the three dimensional field equations of generalized primary creep be satisfied in the interior of the beam, and the condition of zero stress on the lateral boundary hold pointwise. Following the Saint-Venant approximation of elasticity, we require that only the integrals of the end conditions need to be satisfied. There has been other work on creep in beams subject to pure bending, e.g. [3–5]; however, to the best of our knowledge, it has always been from the standpoint of strength of materials, not Saint-Venant theory, and has not treated the problem of *a priori* bounds.

The creep law used in this work is given by (1.3) below. For $m = 0$, it describes secondary creep, while for $m > 0$ it becomes a generalization of the strain-hardening primary creep law (1.6) of Odqvist and Hult [6]. In Section 2, the field equations for small-strain quasistatic creep are given, and the simplifying assumption (2.9) is made for the stresses. This says that the stress field is essentially that of strength of materials, as suggested by the solution of the corresponding elastic problem. We can then derive the nonlinear integral equation (2.24) which is the pure bending analogue‡ of (2.34) in [1]. The unknown effective stress becomes, in this case, simply the magnitude of the tensile stress. In the course of deriving (2.24), it is shown that, in general, a paradox arises unless the elastic response is assumed to be incompressible. This is probably due to the degree of approximation introduced by the assumption (2.9). As in [1], a formal derivation of the limiting stress state is obtained using only the assumption that the terminal moments tend to a finite limit. The result, (2.29), agrees exactly with Hult's result (27) of [5] in the special case where the Odqvist–Hult strain-hardening creep law (1.6) is used. Hult considered only constant moments and made the additional assumption that elastic strains could be neglected in comparison to creep strains.

In Section 3, the tensile stress σ is shown to have the monotonicity properties (3.5), (3.6) and (3.20). These results would seem to be of interest in their own right, since they furnish the qualitative shape of the stress profile, in terms of monotonicity and concavity, as a consequence

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‡A special case of (2.24) was obtained by Goodey [4], eqn (4), for secondary creep in a beam subject to a constant bending moment. A power law with time hardening was used.

of simple analytic arguments. Previously, such information required numerical methods. In fact, the above results would appear to be useful as a check on the validity of numerical output.

Stress bounds (3.9) are easily obtained from (3.5) to (3.6) by means of arguments similar to those used in[1]. It is then shown how these may be reapplied to obtain refined short-time bounds, (3.11) and (3.12), which tend to the exact solution as $t \to 0$. Such bounds may be technologically valuable, since, as remarked by Hult[5], very short creep life-times are now considered in some design work.

The major analytical task of this paper is the derivation of (3.6). For this, we must either restrict ourselves to secondary creep or assume the Odqvist–Hult primary creep power law (1.6). With these restrictions (together with (1.8) and (1.9)), the analogous arguments of [1] would have sufficed. However, the present paper uses a somewhat different and much more efficient argument based on the linear integral eqn (3.15). This approach is especially valuable in working with primary creep, and greatly simplifies the calculations.

As in[1], the infinitesimal strains ϵ_{ij} are assumed to have the form

$$\epsilon_{ij} = \epsilon_{ij}^{(e)} + \epsilon_{ij}^{(c)}, \tag{1.1}$$

where $\epsilon_{ij}^{(e)}$ and $\epsilon_{ij}^{(c)}$ denote elastic strains and creep strains respectively. These are related to the stresses σ_{ij} by the equations†

$$\epsilon_{ij}^{(e)} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}], \tag{1.2}$$

$$\epsilon_{ij}^{(c)}|_{t=0} = 0 \tag{1.3a}$$

$$\dot{\epsilon}_{ij}^{(c)} = \frac{F(\sigma_e)}{[\epsilon_e^{(c)}]^m}\cdot s_{ij},\ t > 0. \tag{1.3b}$$

Here s_{ij} stands for the deviatoric components of the stress, σ_e is the effective stress and $\epsilon_e^{(c)}$ is the effective creep strain. They are defined by the formulas

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\cdot\sigma_{kk}, \tag{1.4}$$

$$\sigma_e = \sqrt{\left(\frac{3}{2}s_{ij}s_{ij}\right)},\quad \epsilon_e^{(c)} = \sqrt{\left(\frac{2}{3}\epsilon_{ij}^{(c)}\epsilon_{ij}^{(c)}\right)}. \tag{1.5}$$

For $m = 0$, (1.3b) gives a generalized secondary creep law; for $m > 0$, (1.3b) generalizes the primary creep law

$$\dot{\epsilon}_{ij}^{(c)} = \frac{3}{2}\frac{K\sigma_e^{n-1}}{[\epsilon_e^{(c)}]^m}\cdot s_{ij} \tag{1.6}$$

of Odqvist and Hult[6].

It is assumed that

$$E > 0,\ -1 < \nu \leqslant \frac{1}{2},\ m \geqslant 0, \tag{1.7}$$

that F is C^2, and that for $z > 0$

$$F(z) > 0, \frac{d}{dz}[zF(z)] > 0, \tag{1.8}$$

$$zF'(z) \geqslant mF(z). \tag{1.9}$$

†Subscripts have the range 1,2,3, δ_{ij} stands for the Kronecker delta, and summation over repeated indices is implied. We shall also use a superposed dot to denote differentiation with respect to time. Points in three-space are designated either (x_1,x_2,x_3) or (x,y,z).

Notice that, in the special case of the creep law (1.6), where

$$F(z) = \frac{3K}{2} z^{n-1}(K > 0), \tag{1.10}$$

(1.9) reduces to

$$n \geq m + 1. \tag{1.11}$$

This condition occurs repeatedly in the primary creep literature [1,2,5].

2. SAINT-VENANT PURE BENDING

As is usual for problems of Saint-Venant type, we assume a cylindrical or prismatic region D of length l with lateral boundary B_L and a uniform cross section which may be represented by a region R in the x,y plane. Coordinates are chosen in such a way that the z axis is parallel to the generators of B_L. Also, it is assumed that R is symmetric with respect to both the x and y axes. It follows that

$$\int_R x \, dA = \int_R y \, dA = \int_R xy \, dA = 0. \tag{2.1}$$

The lateral boundary conditions are

$$\begin{aligned} \sigma_{xx}n_x + \sigma_{xy}n_y &= 0 \\ \sigma_{yx}n_x + \sigma_{yy}n_y &= 0 \\ \sigma_{zx}n_x + \sigma_{zy}n_y &= 0 \end{aligned} \tag{2.2}$$

on $B_L \times [0, \infty)$. On the surface $z = l$, are imposed the relaxed Saint-Venant end conditions

$$\int_R \sigma_{zx} \, dA = \int_R \sigma_{zy} \, dA = \int_R \sigma_{zz} \, dA = 0 \tag{2.3}$$

$$\int_R y\sigma_{zz} \, dA = \int_R (x\sigma_{zy} - y\sigma_{zx}) \, dA = 0 \tag{2.4}$$

$$-\int_R x\sigma_{zz} \, dA = M(t) \tag{2.5}$$

for pure bending in the x,z plane. It is assumed that

$$M(0) \leq 0; M < 0, \dot{M} \leq 0 \; (0 < t < \infty), \tag{2.6}$$

so that the region $x > 0$ is in a state of tension.

The field equations appropriate to this problem are the strain–stress relations of Section 1, the quasistatic equations of motion

$$\sigma_{ij,j} = 0 \tag{2.7}$$

and the strain equations of compatibility (Gurtin [7])

$$\epsilon_{ij,kk} + \epsilon_{kk,ij} - \epsilon_{ik,jk} - \epsilon_{jk,ik} = 0. \tag{2.8}$$

As in the elastic case, we assume a solution of the form

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{zy} = 0, \sigma_{zz} = \sigma(x, t). \tag{2.9}$$

Thus, (2.2), the second equation of (2.4) and (2.7) are immediately satisfied, as well at the first two equations of (2.3). Since R is symmetric with respect to the y-axis, we can satisfy the third equation in (2.3) and the first part of (2.4) by requiring $\sigma(x, t)$ to be an odd function of x.

Assumptions (2.9) also imply that

$$s_{ij} = 0, i \neq j; s_{11} = s_{22} = -\frac{\sigma}{3}, s_{33} = \frac{2\sigma}{3}, \sigma_e = |\sigma|. \tag{2.10}$$

From these relations, together with (2.10), (1.3) and (1.5), it follows that

$$\epsilon_{ij}^{(c)} = 0\,(i \neq j), \epsilon_{11}^{(c)} = \epsilon_{22}^{(c)} = -\frac{1}{2}\epsilon_{33}^{(c)}, \epsilon_e^{(c)} = |\epsilon_{33}^{(c)}|. \tag{2.11}$$

Thus, for $i = j = 3$, (1.3) becomes

$$\dot{\epsilon}_{zz}^{(c)} = \frac{2F(|\sigma|)\sigma}{3|\epsilon_{zz}^{(c)}|^m}, \epsilon_{zz}^{(c)}|_{t=0} = 0.$$

On physical grounds, we restrict ourselves to solutions of the boundary value problem for which

$$\sigma > 0, \epsilon_{zz}^{(c)} > 0$$

in the half-beam $0 < x < c$, where $2c$ is the x-diameter of the cross section. Then, the above initial value problem may be integrated in this region to give

$$\epsilon_{zz}^{(c)} = \left[\int_0^t G(\sigma)\,\mathrm{d}\tau\right]^{1/(m+1)} (x > 0, t \geq 0), \tag{2.12}$$

where

$$G(\sigma) = \frac{2}{3}(m+1)F(\sigma)\sigma. \tag{2.13}$$

Since $\sigma(x, t)$ is odd in x, $\epsilon_{zz}^{(c)}$ may be computed for $x < 0$ by taking its odd extension.

Using (1.1), (1.2), (2.9) and (2.11), we obtain the following strain–stress relations:

$$\epsilon_{xx} = \epsilon_{yy} = -\frac{\nu}{E}\sigma - \frac{1}{2}\epsilon_{zz}^{(c)}, \tag{2.14}$$

$$\epsilon_{zz} = \frac{\sigma}{E} + \epsilon_{zz}^{(c)}, \tag{2.15}$$

$$\epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0. \tag{2.16}$$

Since the shear strains vanish and the principal strains depend only on x, the compatibility eqns (2.8) reduce to

$$\frac{\partial^2}{\partial x^2}[\epsilon_{yy}] = \frac{\partial^2}{\partial x^2}[\epsilon_{zz}] = 0. \tag{2.17}$$

Thus, ϵ_{yy} and ϵ_{zz} are linear in x. This leads to an interesting paradox. For, if we eliminate the creep term between (2.14) and (2.15), we find that

$$\epsilon_{yy} + \frac{1}{2}\epsilon_{zz} = \frac{\sigma}{E}\left(\frac{1}{2} - \nu\right). \tag{2.18}$$

Hence, if $\nu \neq \frac{1}{2}$, σ must be linear in x for all time. This contradicts the fact (shown below) that

$\sigma(x, \infty)$ is generally nonlinear. One might possibly infer from this that, for $\nu \neq \frac{1}{2}$, the assumption (2.9) is at most approximately valid for nonlinear creep.

Since ϵ_{zz} is both linear and odd in x, it must have the form

$$\epsilon_{zz}(x, t) = a(t)\, x, \tag{2.19}$$

so that (2.15) becomes

$$a(t)x = \frac{\sigma}{E} + \epsilon_{zz}^{(c)}. \tag{2.20}$$

Multiplying both sides of (2.20) by x and integrating the resulting equation over R, we use (2.5) to obtain

$$a(t) = -\frac{M(t)}{EI} + \frac{1}{I}\int_R \epsilon_{zz}^{(c)}\, x\, \mathrm{d}A, \tag{2.21}$$

$$I = \int_R x^2\, \mathrm{d}A.$$

Due to the symmetry of R, we can define a nonnegative function $k(x)$, depending only on the geometry of R, such that, for any odd function $f(x)$,

$$\int_R f(x)x\, \mathrm{d}A = \int_0^c f(x)k(x)\, \mathrm{d}x. \tag{2.22}$$

In particular,

$$I = \int_0^c xk(x)\, \mathrm{d}x. \tag{2.23}$$

Using (2.12), (2.20), (2.21) and (2.22), we obtain for $\sigma(x, t)$ the following nonlinear integral equation in the region $[0, c] \times [0, \infty]$:

$$\frac{\sigma(x, t)}{E} = -\frac{M(t)x}{EI} + \frac{x}{I}\int_0^c \left[\int_0^t G(\sigma)\, \mathrm{d}\tau\right]^{1/(m+1)} k(\xi)\, \mathrm{d}\xi - \left[\int_0^t G(\sigma)\, \mathrm{d}\tau\right]^{1/(m+1)}. \tag{2.24}$$

This equation is the pure-bending analogue to (2.34) of [1], on which all the results of that paper were based. Notice that at $t = 0$, we recover the linear elasticity solution. In order to *formally* derive an expression for $\sigma(x, \infty)$, we assume *a priori* that finite limits $\sigma(x, \infty)$, $M(\infty)$ exist. Then after some time t_1, we must have $\sigma(x, t) = \sigma(x, \infty)$ to as many decimal places as is desired. Thus for $t > t_1$ and $0 < x \leq c$,

$$(t - t_1)^{(-1/m+1)}\frac{\sigma(x, t)}{E} = -(t - t_1)^{(-1/m+1)}\frac{M(t)x}{EI} + \frac{x}{I}\int_0^c \left[\frac{\int_0^{t_1} G(\sigma)\, \mathrm{d}\tau}{t - t_1} + G(\sigma(\xi, \infty))\right]^{1/(m+1)} k(\xi)\, \mathrm{d}\xi$$
$$- \left[\frac{\int_0^{t_1} G(\sigma)\, \mathrm{d}\tau}{t - t_1} + G(\sigma(x, \infty))\right]^{1/(m+1)}.$$

Taking the limit as $t \to \infty$, we obtain

$$[G(\sigma(x, \infty))]^{(1/m+1)} = \frac{x}{I}\int_0^c [G(\sigma(\xi, \infty))]^{1/(m+1)} k(\xi)\, \mathrm{d}\xi.$$

This is a linear integral equation in $[G(\sigma(x, \infty))]^{(1/m+1)}$, which has the solution

$$[G(\sigma(x, \infty))]^{1/(m+1)} = Ax,$$

where A is an arbitrary constant. Let $G(\sigma)$ have inverse function H, i.e.

$$H(G(\sigma)) = \sigma.$$

Then,

$$\sigma(x, \infty) = H(A^{m+1}x^{m+1}). \tag{2.25}$$

In order to evaluate A, we multiply (2.24) by $k(x)$ and integrate. We obtain

$$\int_0^c \sigma(x, t)k(x)\,dx = -M(t), \tag{2.26}$$

which is just (2.5) rewritten in the notation of (2.22). Since (2.26) holds for all finite t, it is reasonable that

$$\int_0^c \sigma(x, \infty)k(x)\,dx = -M(\infty). \tag{2.27}$$

Substituting from (2.25) into (2.27), we find that A must satisfy the equation

$$\int_0^c H(A^{m+1}x^{m+1})k(x)\,dx = -M(\infty).$$

In particular, suppose the power law (1.6) of Odqvist and Hult holds. Then

$$G(\sigma) = K(m+1)\sigma^n, H(z) = \left[\frac{z}{K(m+1)}\right]^{(1/n)}, \tag{2.28}$$

and

$$A^{m+1} = \frac{[-M(\infty)]^n K(m+1)}{\left[\int_0^c x^{(m+1)/n}k(x)\,dx\right]^n}.$$

Therefore, by (2.25),

$$\sigma(x, \infty) = \frac{-M(\infty)x^{(m+1)/n}}{\int_0^c \xi^{(m+1)/n}k(\xi)\,d\xi} \tag{2.29}$$

Thus, for the power law (1.6), $\sigma(x, \infty)$ is linear in x only when $m + 1 = n$.

In order to compute the displacements u_x, u_y, u_z, we observe that by (2.14), (2.15) and (2.19), together with the assumption $\nu = \frac{1}{2}$, the nonzero strains take the form

$$\epsilon_{xx} = \epsilon_{yy} = -\frac{1}{2}\epsilon_{zz} = -\frac{a(t)}{2}x, \tag{2.30}$$

where $a(t)$ is given by (2.21). If one assumes that the beam is pinned at $(0, 0, 0)$ in such a way that rigid motions are excluded, the displacements u_x, u_y, u_z are given by

$$u_x = -\frac{a(t)}{2}\left[z^2 + \frac{1}{2}(x^2 - y^2)\right], u_y = -\frac{a(t)}{2}xy, u_z = a(t)xz. \tag{2.31}$$

In particular, the transverse displacement of the center line in the x direction is given by

$$u_x(0, 0, z, t) = -\frac{a(t)}{2}z^2. \tag{2.32}$$

3. BOUNDS AND MONOTONICITY PROPERTIES

The results of this section will be established for C^2 solutions $\sigma(x, t) > 0$ in $(0, c] \times [0, \infty)$. This assumption can be verified rigorously at $t = 0$ and formally at $t = \infty$ (for the power law case) using (2.6) and (2.29). Recall that, by (2.13),

$$G(\sigma) = \frac{2}{3}(m+1)F(\sigma)\sigma, \tag{3.1}$$

and define

$$\phi(x, t) = \int_0^t G(\sigma(x, \tau))\,d\tau = [\epsilon_{zz}^{(c)}]^{m+1}. \tag{3.2}$$

Notice that, by (1.8) and (3.1),

$$G(\sigma) > 0, G'(\sigma) > 0\,(0 < \sigma < \infty). \tag{3.3}$$

With this notation, (2.24) becomes

$$\frac{\sigma(x, t)}{E} = \frac{x}{I}\left(-\frac{M(t)}{E} + \int_0^c \phi^{1/(m+1)}(\xi, t)k(\xi)\,d\xi\right) - \phi^{1/(m+1)}(x, t). \tag{3.4}$$

Our bounding arguments are based on the following inequalities:

$$\frac{\partial \sigma}{\partial x} \geq 0\,(0 < x \leq c, t \geq 0), \tag{3.5}$$

$$\text{(a)}\quad \frac{\partial}{\partial x}\left(\frac{\sigma}{x}\right) \leq 0, \quad \text{(b)}\quad \frac{\partial}{\partial x}\left[\frac{\phi^{1/(m+1)}}{x}\right] \geq 0\,(0 < x \leq c, t \geq 0). \tag{3.6}$$

Due to the assumed oddness and continuity of $\sigma(x, t)$ with respect to x, the lower bound for σ on $[0, c]$ is

$$\sigma(0, t) = 0. \tag{3.7}$$

In order to derive an upper bound, we apply (3.6) (b) to the integral on the right-hand side of (3.4) to obtain

$$\frac{\sigma(x, t)}{E} \leq -\frac{xM(t)}{EI} + \frac{x}{c}\phi^{1/(m+1)}(c, t) - \phi^{1/(m+1)}(x, t). \tag{3.8}$$

Then

$$\sigma(c, t) \leq -\frac{cM(t)}{I},$$

so that (3.5) and (3.7) imply

$$0 \leq \sigma(x, t) \leq -\frac{cM(t)}{I}\,(0 \leq x \leq c, t \geq 0). \tag{3.9}$$

This inequality states, in effect, that the range of values taken by the creep stress at any time t lies within that of the quasistatic elastic stress obtained by setting $F \equiv 0$.

As in [8], the bounds (3.9) may be used to construct additional upper and lower bounds for $\sigma(x, t)$ which are more accurate for short times and which reduce to the exact solution at $t = 0$. In fact, (3.8) and (3.9) imply that

$$\frac{\sigma(x, t)}{E} \leq \frac{-xM(t)}{EI} + \frac{x}{c}\left[\int_0^t G[\sigma(c, \tau)]\,d\tau\right]^{1/(m+1)}. \tag{3.10}$$

Since, by (3.3), G is an increasing function of its arguement, we can apply (3.9)–(3.10) to obtain

$$\frac{\sigma(x,t)}{E} \leqslant \frac{-xM(t)}{EI} + \frac{x}{c}\left[\int_0^t G\left(\frac{-cM(\tau)}{I}\right)\mathrm{d}\tau\right]^{1/(m+1)}. \tag{3.11}$$

Similar reasoning yields the short-time lower bound

$$\frac{\sigma(x,t)}{E} \geqslant \frac{-xM(t)}{EI} - \frac{x}{c}\left[\int_0^t G\left(\frac{-cM(\tau)}{I}\right)\mathrm{d}\tau\right]^{1/(m+1)}. \tag{3.12}$$

The above inequalities may also be used to furnish bounds for the transverse displacement of the center line. For instance, by virtue of (2.21), (2.22), (2.32) and (3.2),

$$u_x(0,0,z,t) = -\frac{z^2}{2}\left(-\frac{M(t)}{EI} + \frac{1}{I}\int_0^c \phi^{1/(m+1)}(\xi,t)k(\xi)\,\mathrm{d}\xi\right).$$

Applying (3.6) (b) and (3.8) to the integral term, we get

$$u_x(0,0,z,t) \geqslant -\frac{z^2}{2}\left(-\frac{M(t)}{EI} + \frac{1}{c}\left[\int_0^t G\left(\frac{-cM(\tau)}{I}\right)\mathrm{d}\tau\right]^{1/(m+1)}\right). \tag{3.13}$$

Also,

$$u_x(0,0,z,t) \leqslant \frac{M(t)z^2}{2EI}. \tag{3.14}$$

The rest of this section is devoted to the proofs of (3.5), (3.6), which are based on the following elementary Lemma. *Let $y(t)$ satisfy the integral equation*

$$y(t) + k_1(t)\int_0^t k_2(\tau)y(\tau)\,\mathrm{d}\tau = f(t) \qquad (t>0), \tag{3.15}$$

where k_1, k_2 and f are nonnegative on $(0,\infty)$. Suppose that k_1 is nonincreasing and f is nondecreasing on $(0,\infty)$. Then y is nonnegative on $(0,\infty)$.

It is assumed that all the quantities involved are suitably smooth functions of t. Clearly, k_1 may be allowed a mild singularity at 0. This lemma can be verified by using the fact that (3.15) has the solution

$$y(t) = f(t) - k_1(t)\int_0^t k_2(\tau)f(\tau)\exp\left[-\int_\tau^t k_1k_2\,\mathrm{d}\lambda\right]\mathrm{d}\tau.$$

In order to apply it to the proof of (3.5), we differentiate (3.4) with respect to x, thus obtaining

$$\frac{1}{E}\frac{\partial\sigma}{\partial x} + \frac{1}{m+1}\phi^{-m/(m+1)}\int_0^t G'(\sigma)\frac{\partial\sigma}{\partial x}\,\mathrm{d}\tau = \frac{1}{I}\left(-\frac{M(t)}{E} + \int_0^c \phi^{1/(m+1)}(\xi,t)k(\xi)\,\mathrm{d}\xi\right). \tag{3.16}$$

This can be thought of as an integral equation in $(\partial\sigma/\partial x)$ of the form (3.15) for which the various hypotheses of the lemma follow from the assumptions (2.6) and (3.3) and the definitions (2.22) (which says $k \geqslant 0$), (2.23) and (3.2).

For the proof of (3.6), we divide both sides of (3.4) by x and differentiate with respect to x. Thus,

$$\frac{1}{E}\frac{\partial}{\partial x}\left(\frac{\sigma}{x}\right) = -\frac{\partial}{\partial x}\left(\frac{\phi^{1/(m+1)}}{x}\right),$$

which shows that (3.6) (b) follows from (3.6) (a). In order to verify the latter, we carry out the differentiation indicated on the right-hand side of the above equation to obtain

$$\frac{1}{E}\frac{\partial}{\partial x}\left(\frac{\sigma}{x}\right) = \frac{\phi^{1/(m+1)}}{x^2} - \frac{1}{x(m+1)}\phi^{-m/(m+1)}\int_0^t G'(\sigma)\frac{\partial\sigma}{\partial x}\,\mathrm{d}\tau.$$

This can be put in the form

$$\frac{1}{E}\frac{\partial}{\partial x}\left(\frac{-\sigma}{x}\right)+\frac{\phi^{-m/(m+1)}}{m+1}\int_0^t G'(\sigma)\frac{\partial}{\partial x}\left(\frac{-\sigma}{x}\right)\mathrm{d}\tau=\psi(x,t),$$

where

$$\psi(x,t)\equiv\frac{\phi^{-m/(m+1)}}{(m+1)x^2}\int_0^t [G'(\sigma)\sigma-(m+1)G(\sigma)]\,\mathrm{d}\tau.$$

This is an integral equation in $\frac{\partial}{\partial x}\left(\frac{-\sigma}{x}\right)$ of the form (3.15). Notice that, due to (3.1), the assumption (1.9) becomes

$$\sigma G'(\sigma)\geqslant(m+1)\,G(\sigma). \tag{3.17}$$

It follows from (3.17) that $\psi\geqslant 0$. Since $\phi^{-m/(m+1)}$ is nonincreasing and $G'\geqslant 0$, the only remaining condition needed for application of the Lemma is that

$$\frac{\partial\psi}{\partial t}\geqslant 0. \tag{3.18}$$

This may be regarded as an additional constitutive assumption on G, or, equivalently, on F. Condition (3.18) is readily verified for two important cases; namely, for $m=0$ (secondary creep) and for the power law (1.6) where the exponents m and n satisfy (1.11). When $m=0$, (3.18) follows from the definition of ψ and (3.17). When the power law holds, we may use (2.28) and (3.2) to write

$$\psi=\frac{(n-m-1)}{(m+1)x^2}\,\phi^{1/(m+1)},$$

from which (3.18) is immediate.

We conclude this section with a final monotonicity result which does not enter into the derivation of the bounds. Let

$$m=0,\ G''\geqslant 0, \tag{3.19}$$

and differentiate the resulting (3.16) with respect to x. Then,

$$\frac{1}{E}\frac{\partial^2\sigma}{\partial x^2}+\int_0^t G'(\sigma)\frac{\partial^2\sigma}{\partial x^2}\,\mathrm{d}\tau=-\int_0^t G''(\sigma)\left[\frac{\partial\sigma}{\partial x}\right]^2\mathrm{d}\tau.$$

Again, we have an integral equation of the form (3.15), this time in $(\partial^2\sigma/\partial x^2)$, and the hypotheses of the Lemma are satisfied due to (3.3) and (3.19). We may therefore conclude that

$$\frac{\partial^2\sigma}{\partial x^2}\leqslant 0\quad(0\leqslant x\leqslant c,\ t>0). \tag{3.20}$$

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